Celestial mechanics in Kerr spacetime

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Abstract. The dynamical parameters conventionally used to specify the orbit of a test particle in Kerr spacetime are the energy E, the axial component of the angular momentum, L_z, and Carter's constant Q. These parameters are obtained by solving the Hamilton-Jacobi equation for the dynamical problem of geodesic motion. Employing the action-angle variable formalism, on the other hand, yields a different set of constants of motion, namely, the fundamental frequencies ω_r, ω_θ and ω_ϕ associated with the radial, polar and azimuthal components of orbital motion. These frequencies, naturally, determine the time scales of orbital motion and, furthermore, the instantaneous gravitational wave spectrum in the adiabatic approximation. In this article, it is shown that the fundamental frequencies are geometric invariants and explicit formulas in terms of quadratures are derived. The numerical evaluation of these formulas in the case of a rapidly rotating black hole illustrates the behaviour of the fundamental frequencies as orbital parameters such as the semi-latus rectum p, the eccentricity e or the inclination parameter θ_- are varied. The limiting cases of circular, equatorial and Keplerian motion are investigated as well and it is shown that known results are recovered from the general formulas.

PACS numbers: 04.20.Jb, 04.30.Db, 04.70.Bw, 95.30.Sf

Submitted to: *Class. Quantum Grav.*

1. Introduction

In Newtonian theory, there are two basic methods of analysing the motion of a test mass under the attraction of a central, spherically symmetric gravitational field: Either Hamiltonian mechanics is applied to solve the problem in a spherical polar coordinate system or, for bound orbits, one may invoke a canonical transformation to action-angle variables. In the latter case, solving the *Hamilton-Jacobi* differential equation yields the integrals of motion and, in particular, *Kepler's third law* which for elliptical orbits, in geometric units, reads [2, 3]

$$M\Omega_{\rm K} = \left(\frac{1-e^2}{p}\right)^{3/2}, \tag{1}$$

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where $\Omega_{\rm K}$ is the angular frequency of orbital revolution, e is the eccentricity of the orbit, $p/(1-e^2)$ its semi-major axis and M is the mass which acts as the source of the central field.

A generalisation of the Hamilton-Jacobi method to the relativistic motion of a test mass in Kerr spacetime, as proposed by Carter [4], shows that the motion is still separable and there is a complete set of constants of motion: The energy E as measured by an observer at spatial infinity, the axial angular momentum component L_z and the Carter constant Q. Whereas E and L_z reflect isometries of the spacetime geometry, Q arises from the separation of the radial and polar components of motion. The resulting first-order equations of motion can be used to compute the particle's worldline and, in limiting cases such as circular and equatorial orbits, to calculate periods of motion.

However, no attempt has been made yet to formulate a relativistic generalisation of action-angle variables for geodesic motion in Kerr spacetime and to calculate the dynamical frequencies of arbitrary bound non-plunging orbits. Exactly these frequencies would be relevant to the computation of gravitational radiation emitted by stellar-mass compact objects in the vicinity of super-massive black holes if a perturbation theory based on the adiabatic approximation is employed. In this approximation one considers the motion as being nearly geodesic under the assumption that the time scale of radiation reaction, $T_{\rm RR}$, is much larger than the dynamical time scales T_r, T_θ and T_ϕ of the three components of orbital motion. Recently, Hughes [5] has raised the question what time scales should, in general, be chosen for T_r, T_θ and T_ϕ, as there is no obvious way of calculating orbital periods from the integrals of motion if the orbit is neither circular nor equatorial.

In the following, it is shown that suitable action-angle variables can be formulated within a fully relativistic framework for general orbits in Kerr spacetime and formulas for dynamical frequencies in terms of numerically solvable integrals are derived.

2. Dynamical equations and integrals of motion

Let us first review the procedure applied by Carter [4] in order to determine the constants of motion. We define the relativistic Hamiltonian for geodesic motion of a particle that possesses no spin in a spacetime with given metric in the absence of an electromagnetic field as [6]

$$H(x^\alpha, p_\beta) = \frac{1}{2} g^{\mu\nu} p_\mu p_\nu. \tag{2}$$

In this equation, the metric components $g^{\mu\nu}$ are considered to be functions of the coordinates x^α and the quantities p_β are the conjugate momenta of the particle associated with these coordinates. Substituting the contravariant components $g^{\mu\nu}$ of the Kerr metric in the Boyer-Lindquist coordinate representation [7], the explicit form of the Hamiltonian is given by

$$H^{\rm (BL)}(x^\alpha, p_\beta) = -\frac{(r^2+a^2)^2 - \Delta a^2 \sin^2\theta}{2\Delta\Sigma}\,(p_t)^2 - \frac{2aMr}{\Delta\Sigma}\,p_t\,p_\phi$$

$$+\frac{\Delta - a^2\sin^2\theta}{2\Delta\Sigma\sin^2\theta}(p_\phi)^2 + \frac{\Delta}{2\Sigma}(p_r)^2 + \frac{1}{2\Sigma}(p_\theta)^2, \tag{3}$$

where $\Delta = r^2 - 2Mr + a^2$ and $\Sigma = r^2 + a^2\cos^2\theta$.

A complete set of constants of motion can be determined if a canonical transformation $\Phi : (x^\alpha, p_\beta) \mapsto (X^\alpha, P_\beta)$ is found such that the Hamiltonian becomes cyclic in all of the new generalised coordinates X^α, i.e., $(H^{(\mathrm{BL})} \circ \Phi^{-1})(X^\alpha, P_\beta) = H^{(\mathrm{cycl})}(P_\beta)$, and the transformed momenta P_β are thus conserved along the worldline of the particle. The generator of such a canonical transformation, $W(x^\alpha, P_\beta)$, is called the *characteristic function* and is determined by the *Hamilton-Jacobi* differential equation

$$g^{\mu\nu}\frac{\partial W}{\partial x^\mu}\frac{\partial W}{\partial x^\nu} + \mu^2 = 0. \tag{4}$$

Solving this equation, yields three constants of integration, E, L_z and Q, apart from $-\mu^2/2$, the value of the Hamiltonian evaluated along the worldline of the particle. The transformed conjugate momenta P_β can be chosen as any functions of the these constants, i.e., $P_\beta = f_\beta(-\mu^2/2, E, L_z, Q)$, assuming that f is bijective and $\mathcal{C}^\infty$, and the generalised coordinates $X^\alpha = \partial W/\partial P_\alpha$ are obtained from Hamilton's equations of motion,

$$\mu\frac{\mathrm{d}X^\alpha}{\mathrm{d}\tau} = \frac{\partial H}{\partial P_\alpha}^{(\mathrm{cycl})} = \nu^\alpha, \tag{5}$$

where each ν^α is a constant. The generalised coordinates are therefore given by $X^\alpha(\tau) = X^\alpha(0) + \nu^\alpha\tau$ if the Hamiltonian is cyclic in all of these coordinates.

For geodesic motion in Kerr spacetime, the constants of motion $E := -p_t$ and $L_z := p_\phi$ are related to the isometries of the metric in the coordinates t and ϕ. E and L_z can be interpreted, respectively, as the particle's energy and axial angular momentum component as seen by an observer at spatial infinity. The third constant of motion, Q, is obtained by separating radial and polar motion and is called *Carter's constant*. In terms of μ, E, L_z and Q, the characteristic function W is given by

$$W = -Et + \int\frac{\sqrt{R}}{\Delta}\,\mathrm{d}r + \int\sqrt{\Theta}\,\mathrm{d}\theta + L_z\phi, \tag{6}$$

where

$$R = \left[(r^2 + a^2)E - aL_z\right]^2 - \Delta[\mu^2r^2 + (L_z - aE)^2 + Q], \tag{7}$$

$$\Theta = Q - \left[(\mu^2 - E^2)a^2 + \frac{L_z^2}{\sin^2\theta}\right]\cos^2\theta. \tag{8}$$

Since $p_\beta = g_{\beta\alpha}\partial W/\partial x^\alpha$, the following set of first-order equations of motion is obtained [4, 6]:

$$\mu\Sigma\frac{\mathrm{d}t}{\mathrm{d}\tau} = \frac{r^2 + a^2}{\Delta}P - a\left(aE\sin^2\theta - L_z\right), \tag{9}$$

$$\mu\Sigma\frac{\mathrm{d}r}{\mathrm{d}\tau} = \pm\sqrt{R}, \tag{10}$$

$$\mu\Sigma\frac{\mathrm{d}\theta}{\mathrm{d}\tau} = \pm\sqrt{\Theta}, \tag{11}$$

$$\mu\Sigma\frac{\mathrm{d}\phi}{\mathrm{d}\tau} = \frac{a}{\Delta}P - aE + \frac{L_z}{\sin^2\theta}, \tag{12}$$

where $P = E(r^2 + a^2) - aL_z$.

If the conjugate momenta are chosen such that the Hamiltonian becomes identically equal to P_0, i.e., $H^{(\mathrm{id})}(P_\beta) = P_0$, then we obtain $\nu^0 = 1$ and $\nu^k = 0$. Substituting the characteristic function into the identities $\partial W/\partial P_0 = \tau + X^0(0)$ and $\partial W/\partial P_k = X^k(0)$ and adjusting the constants $X^\alpha(0)$ in a suitable way, we find the following well-known integrals of motion [4, 6]:

$$\tau - \tau_0 = \int_{r_0}^{r} \frac{r'^2}{\sqrt{R}}\,\mathrm{d}r' + \int_{\theta_0}^{\theta} \frac{a^2\cos^2\theta'}{\sqrt{\Theta}}\,\mathrm{d}\theta', \tag{13}$$

$$t - t_0 = \frac{1}{2}\int_{r_0}^{r} \frac{1}{\Delta\sqrt{R}}\frac{\partial R}{\partial E}\,\mathrm{d}r' + \frac{1}{2}\int_{\theta_0}^{\theta} \frac{1}{\sqrt{\Theta}}\frac{\partial \Theta}{\partial E}\,\mathrm{d}\theta', \tag{14}$$

$$\phi - \phi_0 = -\frac{1}{2}\int_{r_0}^{r} \frac{1}{\Delta\sqrt{R}}\frac{\partial R}{\partial L_z}\,\mathrm{d}r' - \frac{1}{2}\int_{\theta_0}^{\theta} \frac{1}{\sqrt{\Theta}}\frac{\partial \Theta}{\partial L_z}\,\mathrm{d}\theta', \tag{15}$$

and, finally,

$$\int_{r_0}^{r} \frac{\mathrm{d}\theta'}{\sqrt{R}} = \int_{\theta_0}^{\theta} \frac{\mathrm{d}\theta'}{\sqrt{\Theta}}. \tag{16}$$

In general, there are distinct turning points of the radial and polar motions which are asynchronously passed by the particle because $r(\tau)$ and $\theta(\tau)$ are not periodic functions of time. In consequence, equation (14) evaluates to different intervals of coordinate time when integrating over various cycles of radial or polar motion and, for this reason, there is no obvious way of calculating the time scales T_r, T_θ and T_ϕ from the above integrals.

3. The fundamental frequencies

Even though it might seem that there is no useful notion of orbital frequencies if the coordinate functions representing the motion are not periodic, it is still possible to find a representation in which *complete periodicity* becomes manifest. However, such a representation is not found by means of a coordinate transformation in the usual sense, but by virtue of the more general concept of a *canonical transformation* of both spacetime coordinates and conjugate momenta. By its definition, a canonical transformation acts on the cotangential bundle $T^*(\mathcal{M}_{\mathrm{Kerr}})$ of Kerr spacetime, as opposed to a pure coordinate transformation between charts on the basis manifold $\mathcal{M}_{\mathrm{Kerr}}$. In analogy to phase space trajectories in non-relativistic Hamiltonian mechanics, let us consider the locus of all points in $T^*(\mathcal{M}_{\mathrm{Kerr}})$ which are passed by a particle in the course of its motion along a particular orbit. We shall denote the image of that set under the coordinate chart corresponding to Boyer-Lindquist coordinates x^α and their conjugate momenta p_β by $\mathcal{T}_{\mu,E,L_z,Q}$ for a given set of constants of motion.

The basic topological properties of $\mathcal{T}_{\mu,E,L_z,Q}$ in the case of a bound orbit of the first kind can be readily inferred from the equations determining the conjugate momenta,

$$\begin{aligned} p_t &= -E, \qquad & p_\phi &= L_z, \\ \Delta^2 p_r^2 &= R, \qquad & p_\theta^2 &= \Theta. \end{aligned} \tag{17}$$

Firstly, since the orbital revolution progresses continuously in time, $\mathcal{T}_{\mu,E,L_z,Q}$ is not compact in timelike directions. Secondly, the motion is bound both in the r- and in the θ-domain. Moreover, since p_r is solely a function of r and, likewise, p_θ is a function of θ only, both radial and polar motion are of compact support and of the libration type provided that the orbit is *bound* and *stable*. Otherwise, the orbit either plunges and, hence, the particle approaches the horizon towards future timelike infinity or, if the orbit is only marginally stable, it asymptotically approached the inner turning point. Thirdly, the azimuthal motion is a rotation and, of course, it is of compact support as well because $\phi = 0$ is to be identified with $\phi = 2\pi$.

According to a theorem by Arnold [8], that was formulated in the framework of non-relativistic mechanics, the phase space trajectory of a *fully integrable* system of N degrees of freedom is diffeomorphic to the N-torus $\mathcal{T}^N = (\mathcal{S}_1)^N$ whenever the motion of the system is of *compact support.* It seems plausible, that this theorem is applicable to relativistic problems as well if we account for the distinct role of the timelike coordinate which is related to the non-compactness of the worldline of a particle. Accordingly, as a manifestation of the weak equivalence principle, geodesic motion in Kerr spacetime has only three *physical* degrees of freedom because orbits are the same regardless of the mass μ of the particle. Since $\mathcal{T}_{\mu,E,L_z,Q}$ is defined with respect to a particular coordinate system, there is a well defined notion of projecting this set in the directions of the pair of conjugate variables (t, p_t). The resulting projected set, $\hat{\mathcal{T}}_{\mu,E,L_z,Q}$, resides in the six-dimensional space spanned by the spacelike conjugate pairs (r, p_r), (θ, p_θ) and (ϕ, p_ϕ) and corresponds to a phase space. Indeed, from the topological considerations in the previous paragraph, we see that $\hat{\mathcal{T}}_{\mu,E,L_z,Q}$ is diffeomorphic to the three-torus $\mathcal{T}^3$.

Utilising this toroidal topology, let us now define for each spatial component a closed curve $\mathcal{C}_k \subset \hat{\mathcal{T}}_{\mu,E,L_z,Q}$ which circumscribes the k-th torus and can be contracted into a single point with respect to the other tori. If ψ is the coordinate chart on the cotangential bundle producing the Boyer-Lindquist coordinates and its conjugate momenta, then $\psi^{-1}(\mathcal{C}_k)$ is the pre-image of the k-th loop in $T^*(\mathcal{M}_{\mathrm{Kerr}})$. Furthermore, if we note that the *canonical one-form* of Hamiltonian mechanics, $\mathbf{\Theta} = -E\,\mathbf{d}t + p_r\mathbf{d}r + p_\theta\mathbf{d}\theta + L_z\mathbf{d}\phi$, is the one-form corresponding to the relativistic four-momentum $\mathbf{p}$, i.e.,

$$\langle \mathbf{\Theta}, \boldsymbol{p} \rangle = p_\alpha p^\alpha = -\mu^2, \tag{18}$$

then we can define an *action variable* J_k for the k-th spatial component as

$$J_k = \frac{1}{2\pi} \oint_{\psi^{-1}(\mathcal{C}_k)} \mathbf{\Theta}. \tag{19}$$

It might appear that such an action variable ultimately depends on the choice of the closed curve $\mathcal{C}_k$ which is defined with respect to a particular coordinate system and, thus, one would assume that it is not a geometric invariant. However, J_k is independent of the shape of $\mathcal{C}_k$ because $\mathbf{\Theta}$ is *closed* on $\psi^{-1}(\mathcal{T}_{\mu,E,L_z,Q})$, i.e., $\mathbf{d\Theta} = 0$. The closedness of the canonical one-form is shown by substituting the expressions (17) for the conjugate momenta and applying the rules of exterior differentiation§. The generalised Stokes

§ It is actually the separability of the components of motion which renders the canonical one-form

theorem then yields the desired property of J_k [9]. Furthermore, since transformations between coordinate charts preserve topological features, we could have defined the curves $\mathcal{C}_k$ with respect to any coordinate system and then found their pre-images in $T^*(\mathcal{M}_{\text{Kerr}})$. It is only the topological features of $\mathcal{C}_k$ with respect to the three-torus that matters! Therefore we conclude that the action variables (19) are, indeed, geometric invariants. For convenience, however, we shall denote these variables as J_r, J_θ and J_ϕ as if they were specific for the Boyer-Lindquist coordinate system.

Since motion in Kerr spacetime is separable in the coordinates r, θ and ϕ, we may choose each $\mathcal{C}_k$ such that it is located completely within the (x^k, p_k)-plane. Then the action variables can be easily calculated from cyclic integrals over the spatial conjugate momenta in the Boyer-Lindquist coordinate representation:

$$J_r = \frac{1}{2\pi} \oint p_r \, \mathrm{d}r = \frac{1}{2\pi} \oint \frac{\sqrt{R}}{\Delta} \, \mathrm{d}r, \tag{20}$$

$$J_\theta = \frac{1}{2\pi} \oint p_\theta \, \mathrm{d}\theta = \frac{1}{2\pi} \oint \Theta \, \mathrm{d}\theta, \tag{21}$$

$$J_\phi = \frac{1}{2\pi} \oint p_\phi \, \mathrm{d}\phi = L_z. \tag{22}$$

As can be seen from the above expressions, J_r, J_θ and J_ϕ are functions of the mass μ and the three constants of motion E, L_z and Q.

The generator of the transformation to the action variables J_k is given by $W = \int \mathbf{\Theta}$. It is not globally defined, but changes by an amount of $2\pi J_k$ when integrating over the path $\mathcal{C}_k$. As a consequence, if the spatial coordinate x^k goes through N complete cycles and the other coordinates are kept unchanged, then the generalised coordinate $w^k = \partial W / \partial J_k$ associated with J_k changes by $2\pi N$.‖ This result implies that w_k is a phase or *angle variable* and, since $\Delta w^k = \omega_k \Delta\tau$, the constant ω_k can be interpreted as a frequency. Moreover, as a consequence of the geometrically invariant definition of the action variables J_k, the frequencies ω_k specify fundamental properties of the orbital motion independent of any particular coordinate representation. Therefore, they are called the *fundamental frequencies* of the system. However, these frequencies are subject to a straightforward physical interpretation only with respect to the spatial coordinates r, θ and ϕ, when the paths $\mathcal{C}_k$ are correspondingly chosen as cycles of, respectively, radial, polar and azimuthal motion. Accordingly, we shall use the notation ω_r, ω_θ and ω_ϕ for the fundamental frequencies of an orbit in Kerr spacetime.

The standard procedure of determining fundamental frequencies is to find the explicit form of the Hamiltonian in the action-angle representation, $H^{(\text{aa})}$, and to calculate the frequencies from the partial derivatives with respect to the action variables J_k [2]:

$$\mu\omega_k = \frac{\partial H^{(\text{aa})}}{\partial J_k}. \tag{23}$$

being closed on the dynamical submanifold in the cotangential bundle.

‖ Such a change has to be understood as a virtual displacement.

Unfortunately, the definition of the radial and polar action variables in terms of the non-trivial integrals (20) and (21) does not admit an explicit inversion because none of these integrals can be solved analytically. However, as is shown in Appendix A, the derivatives $\partial H^{(\mathrm{aa})}/\partial J_k$ and, hence, the frequencies ω_k can be found even without knowing the functional form of the Hamiltonian $H^{(\mathrm{aa})}$ if the theorem on implicit functions is employed. This procedure results in equations (A.11)–(A.13) of Appendix A and the fundamental frequencies are therefore given by

$$\frac{2\pi}{\mu\omega^r} = 2a^2z_+^2\left[1-\frac{E(k)}{K(k)}\right]X(r_1,r_2)+2Y(r_1,r_2), \tag{24}$$

$$\frac{2\pi}{\mu\omega^\theta} = \frac{4a^2z_+}{\beta}\left[K(k)-E(k)\right]+\frac{4}{\beta z_+}\frac{X(r_1,r_2)}{Y(r_1,r_2)}K(k), \tag{25}$$

$$\frac{2\pi}{\mu\omega^\phi} = 2\pi\frac{a^2z_+^2[K(k)-E(k)]X(r_1,r_2)+K(k)Y(r_1,r_2)}{L_z[\Pi(z_-^2,k)-K(k)]X(r_1,r_2)+K(k)Z(r_1,r_2)}, \tag{26}$$

where r_1 and r_2 are the turning points of radial motion, $z_\pm^2$ are the two roots of the equation $\Theta(z)=0$ when substituting $\cos\theta = z$ in Θ, $k=z_-^2/z_+^2$, $\beta^2=a^2(\mu^2-E^2)$, $K(k)$, $E(k)$ and $\Pi(z_-^2,k)$ are, respectively, the complete elliptical integrals of the first, second and third kind,

$$K(k) = \int_0^{\pi/2}\frac{\mathrm{d}\psi}{\sqrt{1-k\sin^2\psi}}, \tag{27}$$

$$E(k) = \int_0^{\pi/2}\sqrt{1-k\sin^2\psi}\,\mathrm{d}\psi, \tag{28}$$

$$\Pi(z_-^2,k) = \int_0^{\pi/2}\frac{\mathrm{d}\psi}{\left(1-z_-^2\sin^2\psi\right)\sqrt{1-k\sin^2\psi}}, \tag{29}$$

and $X(r_1,r_2)$, $Y(r_1,r_2)$ and $Z(r_1,r_2)$ are radial integrals defined by

$$X(r_1,r_2) = \int_{r_1}^{r_2}\frac{\mathrm{d}r}{\sqrt{R}}, \tag{30}$$

$$Y(r_1,r_2) = \int_{r_1}^{r_2}\frac{r^2\,\mathrm{d}r}{\sqrt{R}}, \tag{31}$$

$$Z(r_1,r_2) = \int_{r_1}^{r_2}\frac{r\left[L_zr-2M(L_z-aE)\right]}{\Delta\sqrt{R}}\,\mathrm{d}r. \tag{32}$$

The above equations for the fundamental frequencies, however, have two deficiencies: Firstly, they are suggestive of ω_k being dependent on the mass μ of the particle which is, of course, not the case. For this reason, we will cast these equations into a dimensionless scale-invariant form, in which μ does not appear anymore. Secondly, the radial functions X, Y and Z are not proper integrals because the integrated functions are divergent at the turning points r_1 and r_2. Fortunately, they are transformed into well-behaved integrals by virtue of the substitution $r=M\tilde{r}=pM/(1+e\cos\chi)$, where χ varies from 0 to 2π as r goes through a complete cycle. Although this substitution originates from the pure Kepler ellipse, it is actually still expedient even for relativistic orbits that may be very different in shape. In analogy to non-relativistic orbits, p

is called the *semi-latus rectum* and e is the *eccentricity* of the orbit. Working out this substitution, yields the following equations for the dimensionless fundamental frequencies $\tilde{\omega}_k = M\omega_k$:

$$\tilde{\omega}_r = \frac{\pi p K(k)}{(1-e^2)\Lambda}, \tag{33}$$

$$\tilde{\omega}_\theta = \frac{\pi \tilde{\beta} z_+ \tilde{X}}{2\Lambda}, \tag{34}$$

$$\tilde{\omega}_\phi = \frac{(\tilde{Z} - \tilde{L}_z \tilde{X})K(k) + \tilde{L}_z \tilde{X}\Pi(z_-^2, k)}{\Lambda}, \tag{35}$$

where the tilde on top of each symbol indicates the corresponding dimensionless quantity (see Appendix B) and the constant Λ is defined by

$$\Lambda = (\tilde{Y} + \tilde{a}^2 z_+^2 \tilde{X})K(k) - \tilde{a}^2 z_+^2 \tilde{X} E(k). \tag{36}$$

The dimensionless radial integrals in the χ-representation are given by

$$\tilde{X} = \mu M \frac{p}{1-e^2} X(r_1, r_2) = \int_0^\pi \frac{\mathrm{d}\chi}{\sqrt{J(\chi)}}, \tag{37}$$

$$\tilde{Y} = \frac{\mu}{M} \cdot \frac{p}{1-e^2} Y(r_1, r_2) = \int_0^\pi \frac{p^2 \, \mathrm{d}\chi}{(1+e\cos\chi)^2 \sqrt{J(\chi)}}, \tag{38}$$

$$\tilde{Z} = \frac{p}{1-e^2} Z(r_1, r_2) = \int_0^\pi \frac{G(\chi)\, \mathrm{d}\chi}{H(\chi)\sqrt{J(\chi)}}, \tag{39}$$

where the functions G, H and J are defined by

$$\begin{aligned} J(\chi) = {} & (1-\tilde{E}^2)(1-e^2) + 2\left(1 - \tilde{E}^2 - \frac{1-e^2}{p}\right)(1+e\cos\chi) \\ & + \left\{(1-\tilde{E}^2)\frac{3+e^2}{1-e^2} - \frac{4}{p} + \left[\tilde{a}^2(1-\tilde{E}^2) + \tilde{L}_z^2 + \tilde{Q}\right]\frac{1-e^2}{p^2}\right\} \\ & \times (1+e\cos\chi)^2, \end{aligned} \tag{40}$$

$$H(\chi) = 1 - \frac{2}{p}(1+e\cos\chi) + \frac{\tilde{a}^2}{p^2}(1+e\cos\chi)^2, \tag{41}$$

$$G(\chi) = \tilde{L}_z - \frac{2(\tilde{L}_z - \tilde{a}\tilde{E})}{p}(1+e\cos\chi). \tag{42}$$

In the above equations, the constants of motion E, L_z and Q are understood as functions of the orbital parameters p, e and $z_- = \cos\theta_-$. Although analytical expressions for these functions can be derived, it is hardly feasible to substitute these expressions for E, L_z and Q. For given orbital parameters, it is much easier to calculate the constants of motion numerically, as explained in Appendix B, and to substitute the resulting values into $G(\chi)$, $H(\chi)$ and $J(\chi)$.

Apart from the three frequencies ω_r, ω_θ and ω_ϕ, a fourth constant of motion is obtained, which is associated with the timelike generalised coordinate $X^0_{(\mathrm{aa})}(\tau) =$

$X^0_{(\mathrm{aa})}(0) + \gamma\tau$ in the action-angle variable representation. According to equation (A.10) of Appendix A, this constant is given by

$$\mu\gamma = \frac{\left[W(r_1,r_2) + a^2 z_+^2 E X(r_1,r_2)\right] K(k) - a^2 z_+^2 E X(r_1,r_2) E(k)}{\left[Y(r_1,r_2) + a^2 z_+^2 X(r_1,r_2)\right] K(k) - a^2 z_+^2 X(r_1,r_2) E(k)}, \tag{43}$$

where the function $W(r_1,r_2)$ is defined by

$$W(r_1,r_2) = \int_{r_1}^{r_2} \frac{r\left[r^3 + a^2 E r - 2a(L_z - aE)\right] \mathrm{d}r}{\Delta\sqrt{R}}. \tag{44}$$

In dimensionless form, γ can be expressed as

$$\gamma = \frac{1}{\Lambda}\left[(\tilde{W} + \tilde{a}^2 z_+^2 \tilde{E}\tilde{X}) K(k) - \tilde{a}^2 z_+^2 \tilde{E}\tilde{X} E(k)\right], \tag{45}$$

where

$$\tilde{W} = \frac{1}{M} \cdot \frac{p}{1-e^2} W(r_1,r_2) = \int_0^\pi \frac{p^2 F(\chi)\,\mathrm{d}\chi}{(1+e\cos\chi)^2 H(\chi)\sqrt{J(\chi)}}, \tag{46}$$

and the function F is defined by

$$F(\chi) = \tilde{E} + \frac{\tilde{a}^2\tilde{E}}{p^2}(1+e\cos\chi)^2 - \frac{2\tilde{a}(\tilde{L}_z - \tilde{a}\tilde{E})}{p^3}(1+e\cos\chi)^3. \tag{47}$$

As for the physical interpretation of γ, we note that the conjugate momentum associated with the generalised coordinate $X^0_{(\mathrm{aa})}$ is $P_0^{(\mathrm{aa})} = p_t = -E$, i.e., the same as in the Boyer-Lindquist representation. For this reason, one would expect that $X^0_{(\mathrm{aa})}$ is in some sense an interval of coordinate time. However, it cannot be equal to $t - t_0$ as equation (14) tells us that the relation between the elapse of proper time τ and the corresponding interval of coordinate time, $t - t_0$, is not linear. On the other hand, there are three particular scales of proper time, τ_r, τ_θ and τ_ϕ, associated with the three fundamental frequencies, ω_r, ω_θ and ω_ϕ. Substituting each of these constants into the expression for $X^0_{(\mathrm{aa})}$, three scales of *coordinate time* are produced:

$$T_k = X^0_{(\mathrm{aa})}(\tau_k) - X^0_{(\mathrm{aa})}(0) = \frac{2\pi\gamma}{\omega_k}. \tag{48}$$

This seems to be a natural definition of the coordinate-time scales associated with the fundamental frequencies. As we shall see in following Section, the above definition, indeed, reproduces the known periods of coordinate time in the limits of, respectively, circular and equatorial motion. Hence, we are led to the conjecture that γ can be interpreted as a *gravitational Lorentz factor* which relates the three dynamical scales of proper time, τ_r, τ_θ and τ_ϕ, to the corresponding intervals of coordinate time, T_r, T_θ and T_ϕ.

In the general case, this conjecture has the following implication: The ratio of time scales associated with two particular components of motion, say, radial and polar motion, is the same regardless of the frame of reference in which these time scales are defined. For example, if an observer at infinity saw the intervals of time T_r and T_θ, he would find that the ratio T_r/T_θ is equal to τ_r/τ_θ. Of course, this is exactly what one would expect, provided that the fundamental frequencies truly arise from invariant properties

of orbital motion. In conclusion, the ratio of time scales should be independent of the frame of reference in use. With respect to coordinate time t, this is inherently ensured by the above definition of T_r, T_θ and T_ϕ.

Ultimately, the significance of the fundamental frequencies lies in the possibility of expanding a dynamical quantity $q(\tau)$ in a Fourier series with harmonics $\omega_{klm} = k\omega_r + l\omega_\theta + m\omega_\phi$ of these frequencies [2]:

$$q(\tau) = \sum_{k,l,m} q_{klm} \exp(\mathrm{i}\omega_{klm}\tau) = \sum_{\vec{k}} q_{\vec{k}} \exp(\mathrm{i}\vec{k} \cdot \vec{w}), \tag{49}$$

where $\vec{k} = (k, l, m)$ and $\vec{k} \cdot \vec{w} = kw^r + lw^\theta + mw^\phi = \omega_{klm}\tau$ according to definition of the action variables w^r, w^θ and w^ϕ. The Fourier coefficients $q_{\vec{k}}$ of the above expansion are given by

$$q_{\vec{k}} = \frac{1}{(2\pi)^3} \int_0^{2\pi} \mathrm{d}w^r \int_0^{2\pi} \mathrm{d}w^\theta \int_0^{2\pi} \mathrm{d}w^\phi \, q(\vec{k} \cdot \vec{w}/\omega_{\vec{k}}) \exp(-\mathrm{i}\vec{k} \cdot \vec{w}). \tag{50}$$

Therefore, dynamical properties of the particle are completely specified by discrete sets of numbers such as $\{q_{\vec{k}}\}_{\vec{k}\in\mathbb{Z}^3}$. This is the very essence of bound geodesic orbital motion in a Kerr spacetime being *quasi-periodic* rather than quasi-chaotic, as was proposed due to the ergodic properties of these orbits.

4. Limiting cases and numerical results

As an important test to support the validity of the fundamental frequencies derived in the previous Section, we will consider two special cases of bound motion in Kerr spacetime: Firstly, the limit of vanishing eccentricity and, secondly, purely equatorial orbits. In particular, we will see that previously derived equations for the periods and the corresponding azimuthal angles of circular orbits [10] can be reproduced from the general formulas. Briefly summarised, the features of orbits in the two limiting cases are as follows:

- Circular orbits ($e = 0$): In Schwarzschild spacetime, Kepler's third law (1) holds exactly for the period of azimuthal revolution with respect to coordinate time, i.e., $T_\phi = 2\pi M\tilde{r}_0^{3/2}$. For circular orbits in Kerr spacetime, there are two distinct frequencies depending on whether the particle co-revolves or counter-revolves with the rotation of the black hole. If the motion is not confined to the equatorial plane, then the particle changes periodically its altitude such that $2\pi/\omega_\theta$ is twice the elapse of proper time between two subsequent passages of the particle through the equatorial plane. The frequencies ω_θ and ω_ϕ are generally incommensurate. As a consequence, the azimuthal angle $\Phi_\theta = 2\pi\omega_\phi/\omega_\theta$ that is accumulated as the particle completes a cycle of polar motion is different from 2π.
- Equatorial orbits ($Q = 0$): As a consequence of the different shape of the potential of radial motion as compared to the Newtonian potential, non-circular orbits do not close on themselves as the particle revolves around the gravitational centre. The total azimuthal angle Φ_r swept out by the particle as it moves from, say, the

apastron at radius $r_{\rm a} = p/(1-e)$ to the periastron at radius $r_{\rm p} = p/(1+e)$ and back to the apastron is given by $\Phi_r = 2\pi\omega_\phi/\omega_r$.

Actually, there is another limit, called polar orbits, for which $L_z = 0$. In this case, the particle crosses the spin axis of the black hole and the inclination parameter is $z_- = 1$. However, it seems unlikely that polar orbits are astrophysically relevant and we only refer to [11] for a general discussion.

4.1. Circular orbits

Orbits of zero eccentricity, $e = 0$, are called circular. In this case, the particle moves on a spherical surface of constant radial coordinate, $r(\tau) = pM = r_0$, within the bounds of the inclination θ given by $\theta_- \leq \theta \leq \pi - \theta_-$. Since the perihelion and aphelion radii coincide, we have $r_{\rm p} = r_{\rm a} = r_0$ and $R(r_0) = 0$. In consequence, the integrals given by equations (30)–(32) and (44) are undefined in the limit of vanishing eccentricity. The corresponding quantities in the χ-representation, however, have well-defined limits because the functions F, G, H and J are constant if $e = 0$. Hence, we define

$$\begin{aligned} X(r_0, r_0) &= \frac{\pi}{\mu r_0 J_{\rm circ}^{1/2}}, \qquad & W(r_0, r_0) &= \frac{\pi r_0 F_{\rm circ}}{H_{\rm circ} J_{\rm circ}^{1/2}}, \\ Y(r_0, r_0) &= \frac{\pi r_0}{\mu J_{\rm circ}^{1/2}}, \qquad & Z(r_0, r_0) &= \frac{\pi M G_{\rm circ}}{r_0 H_{\rm circ} J_{\rm circ}^{1/2}}, \end{aligned} \tag{51}$$

where the constants $F_{\rm circ}$, $G_{\rm circ}$, $H_{\rm circ}$ and $J_{\rm circ}$ are obtained by substituting $e = 0$ in equations (40)–(42) and (47). The frequencies of the angular components of motion are therefore given by

$$\tilde{\omega}_r^{(\rm circ)} = \frac{\tilde{r}_0 J_{\rm circ}^{1/2} K(k)}{(\tilde{r}_0^2 + \tilde{a}^2 z_+^2) K(k) - \tilde{a}^2 z_+^2 E(k)}, \tag{52}$$

$$\tilde{\omega}_\theta^{(\rm circ)} = \frac{\pi \tilde{\beta} z_+}{2[(\tilde{r}_0^2 + \tilde{a}^2 z_+^2) K(k) - \tilde{a}^2 z_+^2 E(k)]}, \tag{53}$$

$$\tilde{\omega}_\phi^{(\rm circ)} = \frac{\tilde{a}(2\tilde{E}\tilde{r}_0 - \tilde{a}\tilde{L}_z)\tilde{\Delta}_0^{-1} K(k) + \tilde{L}_z \Pi(z_-^2, k)}{(\tilde{r}_0^2 + \tilde{a}^2 z_+^2) K(k) - \tilde{a}^2 z_+^2 E(k)}. \tag{54}$$

The existence of a distinct radial frequency even in the case of circular motion indicates that the limit of the integrated azimuthal angle over a cycle of *radial* motion, Φ_r, does not become 2π as the eccentricity approaches zero. Of course, this is well known from equatorial orbits. On the other hand, we can calculate the integrated azimuthal angle Φ_θ as the particle goes through one cycle of *polar* motion by combining the equations of motion (11) and (12). The result is

$$\Phi_\theta = \frac{4}{\tilde{\beta} z_+} \left[\frac{\tilde{a}(2\tilde{E}\tilde{r}_0 - \tilde{a}\tilde{L}_z)}{\tilde{\Delta}_0} K(k) + \tilde{L}_z \Pi(z_-^2, k) \right], \tag{55}$$

where $\tilde{\Delta}_0 = \tilde{r}_0^2 - 2\tilde{r}_0 + \tilde{a}^2$. When comparing this expression for Φ_θ with the quotient of

$\tilde{\omega}_\phi^{(\mathrm{circ})}$ and $\tilde{\omega}_\theta^{(\mathrm{circ})}$, we see that the following well known equation holds:

$$\Phi_\theta = 2\pi \frac{\tilde{\omega}_\phi^{(\mathrm{circ})}}{\tilde{\omega}_\theta^{(\mathrm{circ})}}. \tag{56}$$

Furthermore, calculating the period of coordinate time T_θ for a cycle of polar motion from equation (14), yields

$$T_\theta = \frac{4}{\tilde{\beta} z_+} \left\{ \frac{\tilde{E}(\tilde{r}_0^2 + \tilde{a}^2)\tilde{r}_0^2 - 2\tilde{a}(\tilde{L}_z - \tilde{a}\tilde{E})\tilde{r}_0}{\tilde{\Delta}_0} K(k) + \tilde{a}^2 z_+^2 \tilde{E}[K(k) - E(k)] \right\}. \tag{57}$$

If the constant γ given by equation (45) is also calculated in the limit $e = 0$, we find that

$$T_\theta = \frac{2\pi\gamma^{(\mathrm{circ})}}{\omega_\theta^{(\mathrm{circ})}}, \tag{58}$$

which suggests that $\gamma^{(\mathrm{circ})}$ can be interpreted as the *Lorentz factor* of circular motion in Kerr spacetime.

4.2. Equatorial orbits

If $Q = 0$, then the equation for the turning points of polar motion, $\Theta(z) = 0$, admits the solutions $z_-^2 = 0$ and $z_+^2 = 1 + L_z^2/\beta^2 > 1$. As a consequence, the motion of the particle is confined to the equatorial plane, i.e., $\theta(\tau) = \pi/2$. Since $k = (z_-/z_+)^2 = 0$, the elliptical integrals become $K(0) = E(0) = \Pi(0, 0) = \pi/2$. The resulting expressions for the fundamental frequencies of equatorial orbits are thus given by

$$\omega_r^{(\mathrm{eq})} = \frac{\pi p}{(1 - e^2)\tilde{Y}_{\mathrm{eq}}}, \tag{59}$$

$$\omega_\theta^{(\mathrm{eq})} = \tilde{\beta} z_+ \frac{\tilde{X}_{\mathrm{eq}}}{\tilde{Y}_{\mathrm{eq}}}, \tag{60}$$

$$\omega_\phi^{(\mathrm{eq})} = \frac{\tilde{Z}_{\mathrm{eq}}}{\tilde{Y}_{\mathrm{eq}}}. \tag{61}$$

As in the case of circular orbits, we still have three different frequencies.

Combining the equations of motion (10) and (12), we can calculate the integrated azimuthal angle Φ_r between two successive passages through, say, the periastron and we find that

$$\Phi_r = 2Z_{\mathrm{eq}} = \frac{2(1 - e^2)}{p} \tilde{Z}_{\mathrm{eq}}. \tag{62}$$

According to equation (14), the corresponding period of coordinate time is given by $T_r = 2W_{\mathrm{eq}}$. Using $\gamma^{(\mathrm{eq})} = \tilde{W}_{\mathrm{eq}}/\tilde{Y}_{\mathrm{eq}}$, we obtain

$$T_r = \frac{2\pi\gamma^{(\mathrm{eq})}}{\omega_r^{(\mathrm{eq})}} \tag{63}$$

and one can see that $\gamma^{(\mathrm{eq})}$ acts as a Lorentz factor which relates the elapse of proper time over one cycle of equatorial motion with the corresponding period of coordinate time.

4.3. Keplerian orbits

In the Newtonian limit, the constants of motion asymptotically approach the values of Keplerian orbits which are given by equations (B.26) in Appendix B. Since the Keplerian values of energy and angular momentum, naturally, are independent of the spin of the black hole, the problem reduces to that one of motion in Schwarzschild spacetime and we can set $z_- = 0$. Thus, we obtain the the asymptotic expressions

$$\begin{aligned} F(\chi) &\simeq 1 + \mathrm{O}(p^{-1}), & G(\chi) &\simeq p^{1/2} + \mathrm{O}(p^{-1/2}), \\ H(\chi) &\simeq 1 + \mathrm{O}(p^{-1}), & J(\chi) &\simeq (1-e^2)^2 p^{-1} + \mathrm{O}(p^{-2}) \end{aligned} \tag{64}$$

which lead to

$$\tilde{W} \simeq \tilde{Y}, \qquad \tilde{X} \simeq \frac{\pi p^{1/2}}{1-e^2}, \qquad \tilde{Y} \simeq \frac{\pi p}{(1-e^2)\Omega_{\mathrm{K}}}, \qquad \tilde{Z} \simeq p^{1/2} \tilde{X}, \tag{65}$$

where Ω_{K} is the Keplerian frequency given by equation (1).

Substitution of the above asymptotic expressions into equations (33)–(35) and (45) results in

$$\gamma \simeq 1, \qquad \tilde{\omega}_r \simeq \tilde{\omega}_\theta \simeq \tilde{\omega}_\phi \simeq \Omega_{\mathrm{K}}. \tag{66}$$

The three components of motion therefore become commensurate in the Newtonian limit and the corresponding frequencies asymptotically degenerate. In the nearly Newtonian regime, where the three frequencies are close to Ω_{K} yet slightly different from each other, we expect that there will be a minute perihelion drift (due to the small difference between ω_ϕ and ω_r) as well as a slow precession of the orbital plane (due to the difference between ω_ϕ and ω_θ). In order to calculate this precession, an approximation of the fundamental frequencies including terms of next lower order in p would be required. However, this is left for future work.

4.4. Numerical case study of a rapidly rotating black hole

Let us now consider bound orbits around a nearly extreme Kerr black hole of spin $\tilde{a} = 0.998$.¶ Such a black hole is particularly interesting because effects due to spacetime dragging caused by the high angular momentum of the black hole are strong and stable orbits exist even very close to the horizon (which is located at $r = r_{\mathrm{H}} \approx 1.0632M$). Subsequently, we will explore the parameter space in all three dimensions and analyse the behaviour of the fundamental frequencies as each of the orbital parameters is varied. The numerical evaluation of equations (33)–(35) and a general algorithm to calculate the constants of motion E, L_z and Q for arbitrary orbital parameters p, e and θ_- was implemented in MATHEMATICA.+

¶ The chosen value of the spin is presumably a critical value as black holes which are accreting material tend to be buffered against further increase of spin if $\tilde{a} \approx 0.998$ [12].

+ The procedures were tested, for instance, by comparison of large numerical samples to the results of Wilkins [13] for circular orbits around an extreme Kerr black hole and those of Cutler *et al* [14] for elliptical orbits in Schwarzschild spacetime. They are available from the author on request.

To begin with, we choose $e = 1/3$ (in this case the aphelion radius is twice the perihelion radius) and $\theta_- = \pi/3$ (corresponding to 30° of maximum altitude from the equatorial plane). The fundamental frequencies of the prograde and retrograde orbits as functions of p are shown in figure 1. Basically, one can see that the frequencies $\tilde{\omega}_r$, $\tilde{\omega}_\phi$ and $\tilde{\omega}_\theta$ increasingly deviate from each other as the distance from the black hole

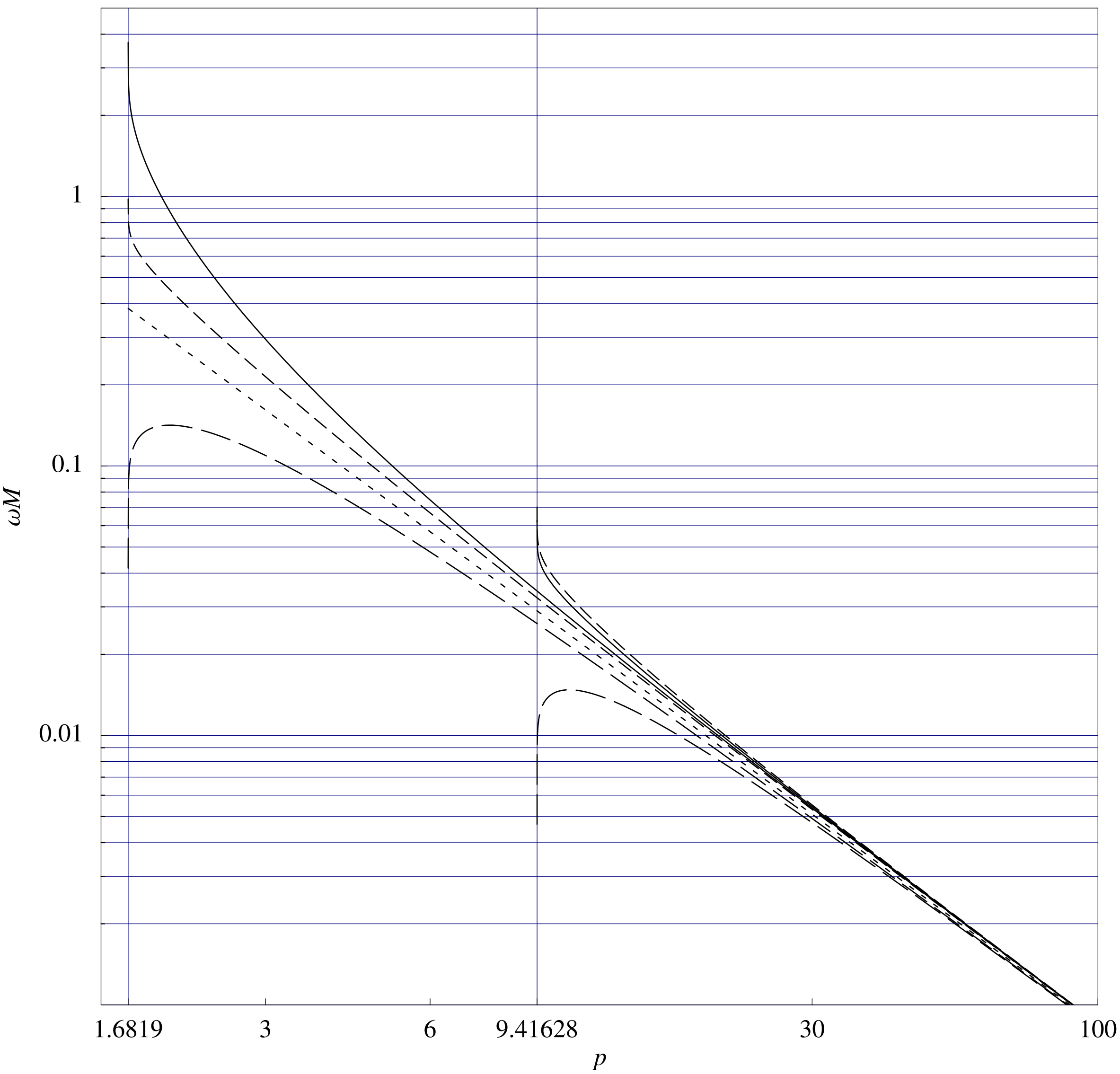

Figure 1. Dimensionless fundamental frequencies $\tilde{\omega}_r$, $\tilde{\omega}_\theta$ and $\tilde{\omega}_\phi$ as functions of p for orbits of eccentricity $e = 1/3$ and minimal inclination $\theta_- = \pi/3$ to the spin axis of the black hole (long-dashed lines: $\tilde{\omega}_r$, short-dashed lines: $\tilde{\omega}_\theta$, solid lines: $\tilde{\omega}_\phi$). The set of lines diverging towards $p \approx 9.416$ corresponds to retrograde orbits (lower binding energy, smaller angular momentum), whereas the other set of lines which diverge towards $p \approx 1.682$ are the frequencies of prograde orbits (higher binding energy, larger angular momentum). The two vertical grid lines mark the locations of the marginally stable prograde and retrograde orbits. The widely spaced dashed straight line is the "asymptotic Keplerian branch" which is given by the power law $\Omega_{\mathrm{K}} = (16\sqrt{2}/27)p^{-3/2}$.

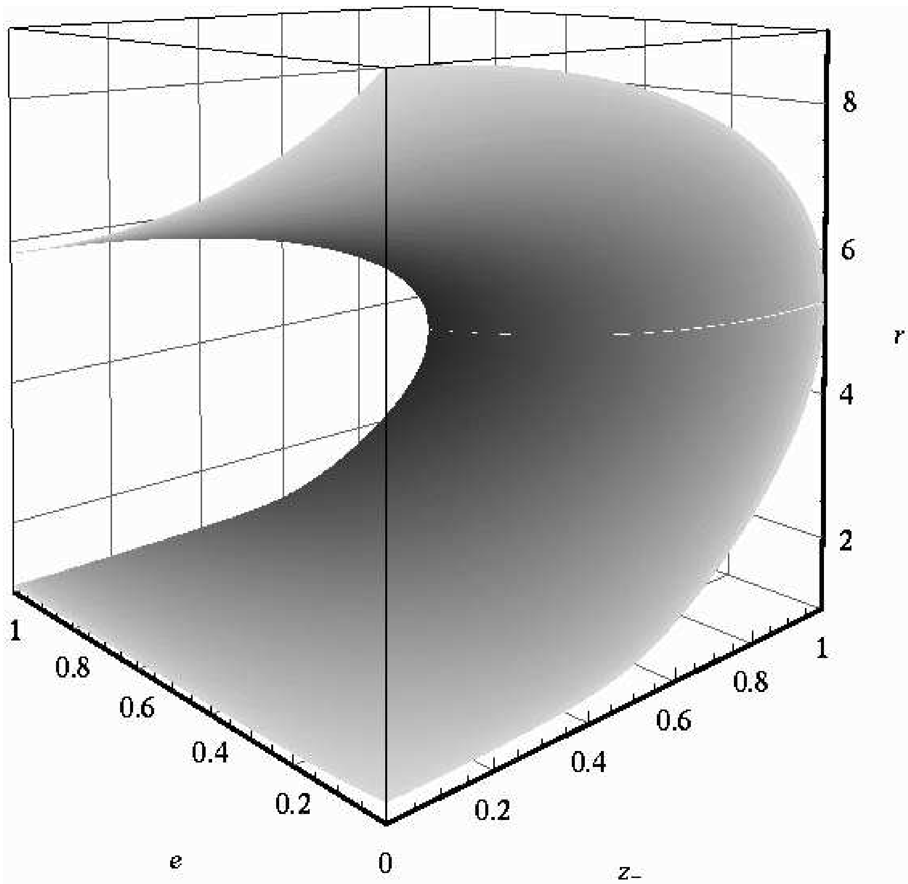

Figure 2. Separatrix of a Kerr black hole of spin $\tilde{a} = 0.998$. The two surfaces specify the smallest radial distance r which can be reached by particles moving in a stable orbit of given eccentricity e and inclination parameter z_- without plunging into the black hole (a high-quality version of this graph can be obtained from the author).

becomes smaller. For increasing p, on the other hand, the fundamental frequencies tend to degenerate and, in fact, they converge towards the Keplerian frequency $\Omega_{\rm K} = (16\sqrt{2}/27)p^{-3/2}$ for $e = 1/3$. Thus, we call the Keplerian power law for given eccentricity the *asymptotic Keplerian branch* of the fundamental frequencies.

The values of p, for which the frequencies of, respectively, prograde and retrograde orbits diverge, are located exactly where the respective orbits become *marginally stable*. In this case, stable prograde orbits only exist if $p > p_{\rm ms}^{\rm (p)} \approx 1.682$ and stable retrograde orbits only if $p > p_{\rm ms}^{\rm (r)} \approx 9.416$. Hence, the corresponding minimal perihelion radii of stable prograde and retrograde orbits with $e = 1/3$ and $\theta_- = \pi/3$ are, respectively, $r_{\rm p}^{\rm (p)} \approx 1.261M$ and $r_{\rm p}^{\rm (r)} \approx 7.062M$. Note that $\tilde{\omega}_r \to 0$ and $\tilde{\omega}_\phi \to \infty$ as $p \to p_{\rm ms}$. This is a consequence of the orbits becoming "zoom whirls" which means that the particle moves from the apastron inwards and then revolves many times along an almost circular spiral before it actually reaches the periastron.* In the limit of a marginally stable orbit, the

* Equatorial zoom whirl orbits have been recently investigated [15].

periastron is approached asymptotically. The elapse of proper time as the particle moves from apastron to periastron, thus, ever increases as $p \to p_{\rm ms}$ and the radial frequency approaches zero. If the particle represents an astrophysical object, however, radiation reaction will finally act on a time scale that is comparable to the dynamical time scale of radial motion ($\sim 2\pi\gamma/\omega_r$) and, as a consequence, it will drive the particle within finite time over the stability threshold so that it plunges into the black hole.

In parameter space, the locus of all points corresponding to marginally stable orbits, $p = p_{\rm ms}(e, z_-)$, is called the *separatrix*. A plot of the separatrix for $\tilde{a} = 0.998$ is shown

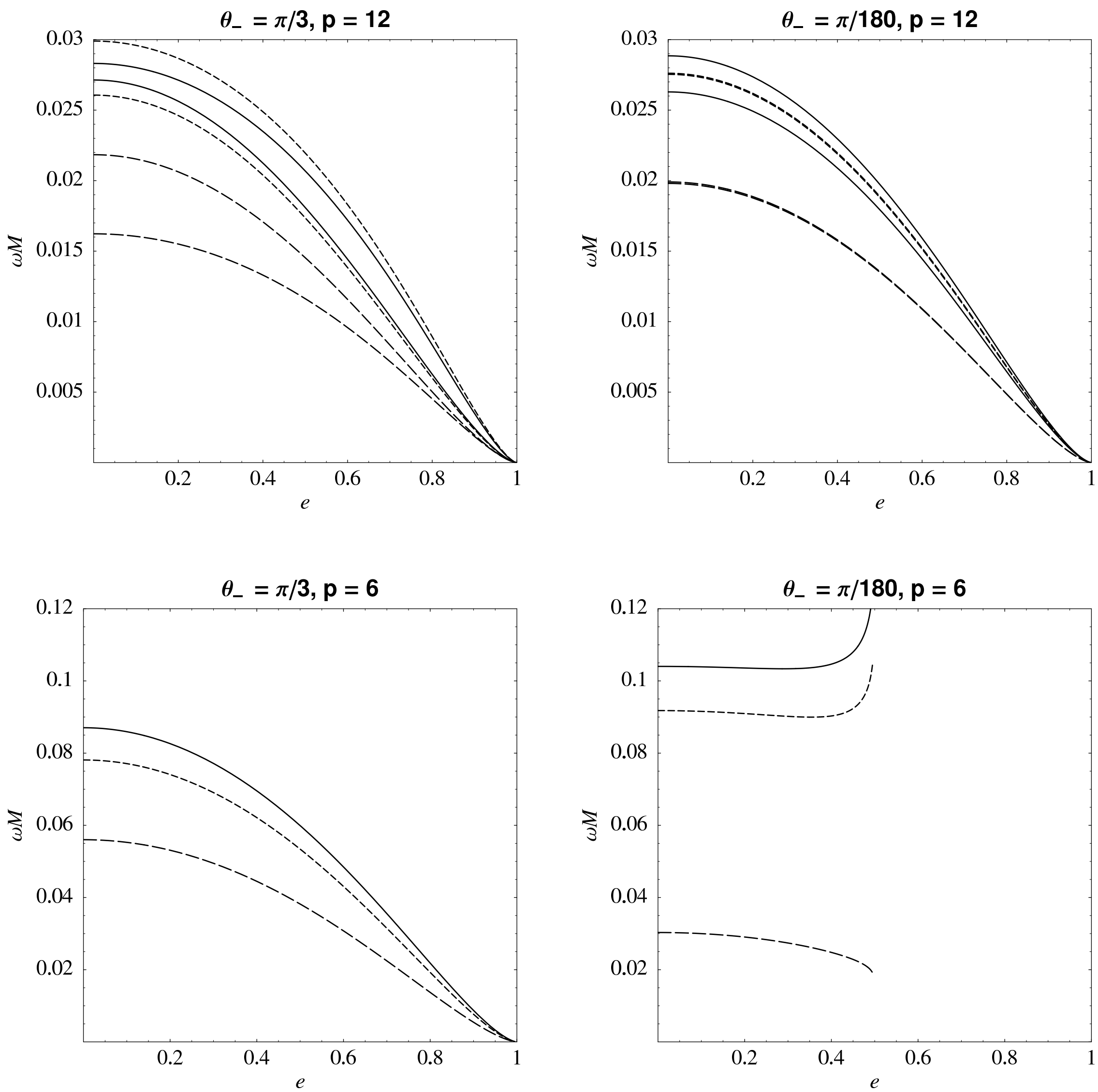

Figure 3. Dimensionless fundamental frequencies $\tilde{\omega}_r$, $\tilde{\omega}_\theta$ and $\tilde{\omega}_\phi$ as functions of the eccentricity e for different values of p and θ_- (long-dashed lines: $\tilde{\omega}_r$, short-dashed lines: $\tilde{\omega}_\theta$, solid lines: $\tilde{\omega}_\phi$; the frequencies of retrograde orbits enclose those of prograde orbits). In the bottom panels, only the frequencies of prograde orbits are plotted since there are no stable retrograde orbits if $p = 6$.

in figure 2, where the perihelion radius $r_\mathrm{p} = p/(1+e)$ is used as parameter in place of p. There are two sheets, one corresponding to prograde marginally stable orbits and the other one corresponding to retrograde marginally stable orbits. The two sheets smoothly join at $z = 1$, i.e., where the polar orbits with $\tilde{L}_z = 0$ are located. As $z_- = \cos\theta_-$ becomes smaller, however, there is an increasing space between the two sheets, in which only prograde but no retrograde orbits reside. As the eccentricity increases, both prograde and retrograde orbits exist closer to the horizon due to the

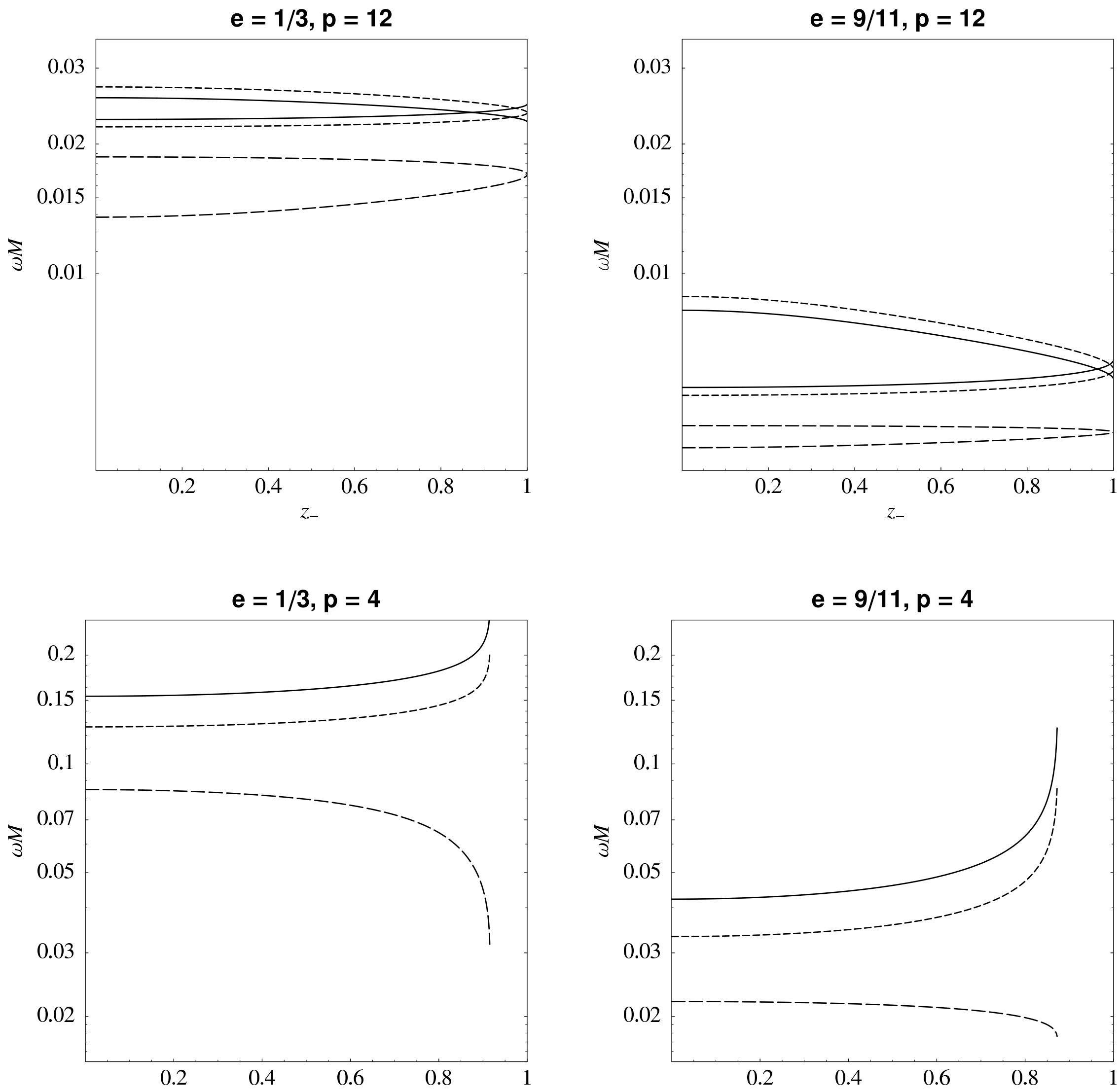

Figure 4. Dimensionless fundamental frequencies $\tilde{\omega}_r$, $\tilde{\omega}_\theta$ and $\tilde{\omega}_\phi$ as functions of the minimal inclination to the spin axis of the black hole, θ_-, for different values of e and p. (long-dashed lines: $\tilde{\omega}_r$, short-dashed lines: $\tilde{\omega}_\theta$, solid lines: $\tilde{\omega}_\phi$; the frequencies of retrograde orbits enclose those of prograde orbits). In the bottom panels, only the frequencies of prograde orbits are plotted since stable retrograde orbits with $p = 4$ do not exist at all. Even the prograde orbits are only stable if the altitude from the equatorial plane is lower than roughly $3\pi/8$ for $e = 1/3$ and about $\pi/3$ for $e = 9/11$.

higher momentum of the particle at the periastron as compared to circular orbits of the same radius. In particular, there are prograde orbits of eccentricity $e \gtrsim 0.5$ with $z \lesssim 0.5$ for which the periastron is very close to the horizon (within $\delta\tilde{r} \sim 0.1$).

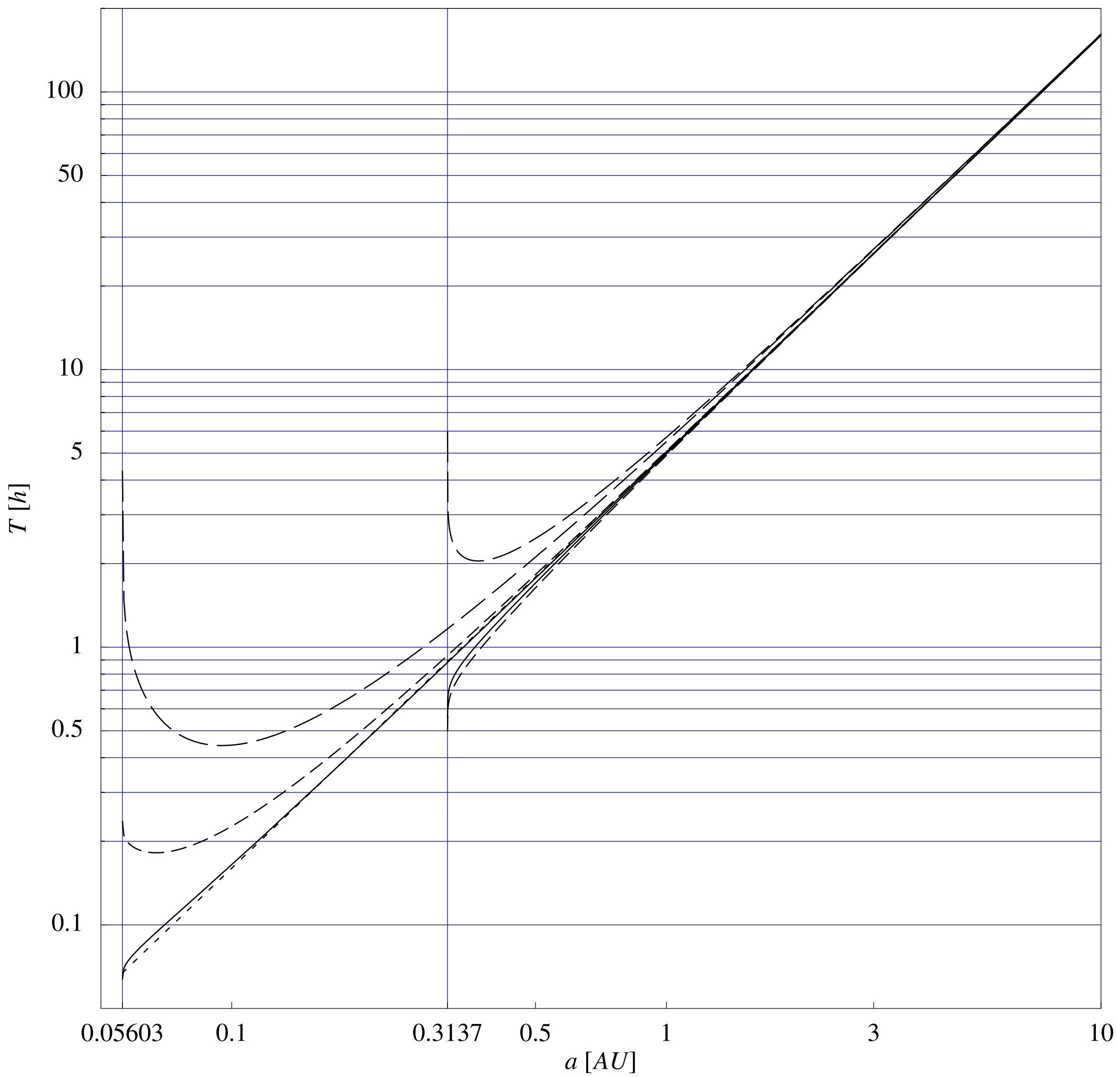

Figure 5. Scales of coordinate time T_r, T_θ and T_ϕ in hours as functions of the semi-major axis $a = pM/(1-e^2)$ in astronomical units ($1\,\mathrm{AU} = 1.496 \cdot 10^{13}\,\mathrm{cm}$) for a central black hole of mass $M = 3 \cdot 10^6 M_{\mathrm{sol}}$ (long-dashed lines: T_r, short-dashed lines: T_θ, solid lines: T_ϕ). As in figure 1, the eccentricity of the orbits is $e = 1/3$ and the inclination parameter $\theta_- = \pi/3$. The set of lines diverging towards $a \approx 0.3137\,\mathrm{AU}$ corresponds to retrograde orbits, whereas the other set of lines which diverge towards $a \approx 0.0560\,\mathrm{AU}$ are the frequencies of prograde orbits. The two vertical grid lines mark the locations of the marginally stable prograde and retrograde orbits. The widely spaced dashed straight line is the Keplerian orbital period given by $P_{\mathrm{K}}^2 = 4\pi^2 a^3/GM$ (in physical units).

Figure 3 shows the change of the fundamental frequencies with eccentricity for various choices of θ_- and p. The main feature of these plots is the convergence of the frequencies to a single value as $e \to 1$. This simply reflects the fact that orbits

of eccentricity very close to unity are almost unbound and the apastron is located far away from the region of strong gravity. In consequence, the particle is mostly moving in the nearly Newtonian regions of spacetime and the fundamental frequencies tend to degenerate. In the plot showing the frequencies of nearly polar orbits with $\theta_- = \pi/180$ and $p = 6$, the frequencies terminate just below $e = 0.5$ as prograde orbits of higher eccentricity do not exist if $z \sim 1$ (note that the value of the perihelion radius depends on e according to $\tilde{r}_\mathrm{p} = 6/(1+e)$ when comparing to the plot of the separatrix in figure 2).

The dependence of the fundamental frequencies on the inclination parameter θ_- (which is the smallest angle of inclination to the spin axis of the black hole that can be reached by the particle) is shown in figure 4 for different values of e and p. Apparently, both the radial and polar frequencies of prograde and retrograde orbits converge towards each other as $\theta_- \to 0$, i.e., in the polar limit. On the other hand, there is a crossing-over of the azimuthal frequencies at some finite value of θ_-. As a consequence, the azimuthal motion of nearly polar retrograde orbits is *faster* than that of the corresponding prograde orbits and, thus, it seems that nearly polar orbits are markedly different from nearly equatorial orbits. We have to keep in mind, however, that the revolution of the particle around the black hole in the case $\theta_- \sim 0$ is rather specified by the polar frequency ω_θ, whereas ω_ϕ determines an azimuthal precession around the spin axis of the black hole.

Finally, in figure 5, the scales of coordinate time, T_r, T_θ and T_ϕ, defined by equation 48 are shown in physical units as functions of the semi-major axis a for orbits of eccentricity $e = 1/3$ and inclination parameter $\theta_- = \pi/3$. The mass of the black hole is chosen to be $M = 3 \cdot 10^6 M_\mathrm{sol}$ which is about the mass determined for the black hole in the centre of our Galaxy [16]. It is interesting to note that for virtually all prograde orbits the time scale of polar motion T_ϕ is very close to the the Keplerian orbital period $P_\mathrm{K} = 2\pi/\Omega_\mathrm{K}$. However, this is only accidentally the case for the chosen set of parameters, as for lower eccentricity T_ϕ is above P_K, whereas it tends to become significantly smaller than P_K at higher eccentricity.

5. Conclusion

The investigation of bound geodesic orbits in Kerr spacetime presented in this article clearly illustrates that the properties of these orbits in the regime of strong gravity are profoundly different from Keplerian orbits in the Newtonian regime. In particular, there are three different frequencies of orbital motion, ω_r, ω_θ and ω_ϕ, which increasingly deviate from the Keplerian frequency as the size of the orbit decreases and finally approaches the limit of a marginally stable orbit. This is a consequence of the radial, polar and azimuthal components of motion being *incommensurate* in Kerr spacetime. As a consequence, the shapes of orbits in the vicinity of a Kerr black hole are exceedingly complex.

The explicit formulas for the fundamental frequencies ω_r, ω_θ and ω_ϕ presented in Section 3 are exact in the test particle limit and useful approximations if radiation reaction is *adiabatic*, i.e., if the time scales associated with the frequencies are small

compared to the time scale of radiation reaction. Based on the constant of motion γ which is associated with a timelike generalised coordinate, we have proposed a conversion of the invariant fundamental frequencies into coordinate-time scales, T_r, T_ϕ and T_θ, which should be relevant to an observer at spatial infinity and, thus, gravitational wave astronomy. Once these time scales are established, one can compute *instantaneous* gravitational wave forms in the adiabatic approximation. In this context, the term "instantaneous" means an interval of time, over which no significant changes of the constants of motion are accumulated due to radiation reaction. From the resulting spectrum, an estimation of the power emitted in the frequency band of a detector could be made, without making use of an advanced theory of radiation reaction.

Acknowledgements

I thank C. Glampedakis for many discussions and making me aware of the problem concerning the dynamical time scales of geodesic orbits in Kerr spacetime and also B. S. Sathyaprakash and L. P. Grishchuk for their support and useful criticism. This work was funded by a *Wilhelm-Macke mobility scheme award* of the Johannes-Kepler-University Linz, Austria, as part of my postgraduate studies at Cardiff University.

Appendix A. Determination of the fundamental frequencies

Let $P^{(\mathrm{aa})}_\beta = f^{(\mathrm{aa})}_\beta(-\mu^2/2, E, L_z, Q)$ be the momenta given by $P^{(\mathrm{aa})}_0 = p_t = -E$ and $P^{(\mathrm{aa})}_k = J_k$, where the J_k are the action variables defined by equations (20)–(22). If we denote the Jacobian matrix of f by $\mathsf{D}f$, then, by the theorem on implicit functions, $\mathsf{D}f \cdot \mathsf{D}(f^{-1}) = \mathsf{D}f \cdot (\mathsf{D}f)^{-1} = \mathsf{I}$, provided that f is non-zero and the Jacobian does not vanish [17]. Since $-\mu^2/2$ is the invariant value of the Hamiltonian, we can substitute $-\mu^2/2 = H^{(\mathrm{aa})}(-E, J_k)$, where $H^{(\mathrm{aa})}$ is the Hamiltonian in the action-angle variable representation. For brevity, let us subsequently use the symbol H to denote the Hamiltonian in that representation. Moreover, two rows of the Jacobian matrix are trivial due to the identities $P^{(\mathrm{aa})}_0 = -E$ and $J_\phi = L_z$. Thus, the equation $\mathsf{D}f \cdot \mathsf{D}(f^{-1}) = \mathsf{I}$ reads

$$\begin{pmatrix} 0 & -1 & 0 & 0 \\ \frac{\partial J_r}{\partial H} & \frac{\partial J_r}{\partial E} & \frac{\partial J_r}{\partial L_z} & \frac{\partial J_r}{\partial Q} \\ \frac{\partial J_\theta}{\partial H} & \frac{\partial J_\theta}{\partial E} & \frac{\partial J_\theta}{\partial L_z} & \frac{\partial J_\theta}{\partial Q} \\ 0 & 0 & 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} -\frac{\partial H}{\partial E} & \frac{\partial H}{\partial J_r} & \frac{\partial H}{\partial J_\theta} & \frac{\partial H}{\partial J_\phi} \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{\partial Q}{\partial E} & \frac{\partial Q}{\partial J_r} & \frac{\partial Q}{\partial J_\theta} & \frac{\partial Q}{\partial J_\phi} \end{pmatrix} = \mathsf{I}. \tag{A.1}$$

The above matrix equation can be split into four non-trivial sets of linear equations in the eight unknowns $-\frac{\partial H}{\partial E}$, $\frac{\partial H}{\partial J_k}$, $-\frac{\partial Q}{\partial E}$ and $\frac{\partial Q}{\partial J_k}$:

$$-\mathsf{A} \cdot \frac{\partial}{\partial E} \begin{pmatrix} H \\ Q \end{pmatrix} = \begin{pmatrix} 2 \int_{r_1}^{r_2} \frac{\mathrm{d}r}{\sqrt{R}} \left[\frac{(r^2+a^2)P}{\Delta} + a(L_z - aE) \right] \\ 2a^2 E \int_{\theta_-}^{\pi/2} \frac{\cos^2\theta \mathrm{d}\theta}{\sqrt{\Theta}} \end{pmatrix} \tag{A.2}$$

$$\mathsf{A}\cdot\frac{\partial}{\partial J_r}\begin{pmatrix} H \\ Q \end{pmatrix}=\begin{pmatrix} 2\pi \\ 0 \end{pmatrix}, \tag{A.3}$$

$$\mathsf{A}\cdot\frac{\partial}{\partial J_\theta}\begin{pmatrix} H \\ Q \end{pmatrix}=\begin{pmatrix} 0 \\ \frac{\pi}{2} \end{pmatrix}, \tag{A.4}$$

$$\mathsf{A}\cdot\frac{\partial}{\partial J_\phi}\begin{pmatrix} H \\ Q \end{pmatrix}=\begin{pmatrix} 2\int_{r_1}^{r_2}\frac{\mathrm{d}r}{\sqrt{R}}\left[\frac{aP}{\Delta}+(L_z-aE)\right] \\ 2L_z\int_{\theta_-}^{\pi/2}\frac{\cot^2\theta\mathrm{d}\theta}{\sqrt{\Theta}} \end{pmatrix} \tag{A.5}$$

where $P=(r^2+a^2)E-aL_z$, and the coefficient matrix A is given by

$$\mathsf{A}=\begin{pmatrix} 2\int_{r_1}^{r_2}\frac{r^2\mathrm{d}r}{\sqrt{R}} & -\int_{r_1}^{r_2}\frac{\mathrm{d}r}{\sqrt{R}} \\ 2a^2\int_{\theta_-}^{\pi/2}\frac{\cos^2\theta\,\mathrm{d}\theta}{\sqrt{\Theta}} & \int_{\theta_-}^{\pi/2}\frac{\mathrm{d}\theta}{\sqrt{\Theta}}. \end{pmatrix} \tag{A.6}$$

Using the definitions (30)–(32) and (44) for the radial integrals and the identities

$$\int_{\theta_-}^{\pi/2}\frac{\mathrm{d}\theta}{\sqrt{\Theta}}=\frac{1}{\beta z_+}K(k), \tag{A.7}$$

$$\int_{\theta_-}^{\pi/2}\frac{\cos^2\theta}{\sqrt{\Theta}}\,\mathrm{d}\theta=\frac{z_+}{\beta}[K(k)-E(k)], \tag{A.8}$$

$$\int_{\theta_-}^{\pi/2}\frac{\cot^2\theta}{\sqrt{\Theta}}\,\mathrm{d}\theta=\frac{z_+}{\beta}[\Pi(z_-^2,k)-K(k)], \tag{A.9}$$

where the elliptical integrals $K(k)$, $E(k)$ and $\Pi(-z_-^2,k)$ are defined by equations (27)–(29), the solutions of the above systems of equations for $-\frac{\partial H}{\partial E}$ and $\frac{\partial H}{\partial J_k}$ are given by

$$-\frac{\partial H}{\partial E}=\frac{K(k)W(r_1,r_2)+a^2z_+^2E\left[K(k)-E(k)\right]X(r_1,r_2)}{K(k)Y(r_1,r_2)+a^2z_+^2\left[K(k)-E(k)\right]X(r_1,r_2)}, \tag{A.10}$$

$$\frac{\partial H}{\partial J_r}=\frac{\pi K(k)}{K(k)Y(r_1,r_2)+a^2z_+^2\left[K(k)-E(k)\right]X(r_1,r_2)}, \tag{A.11}$$

$$\frac{\partial H}{\partial J_\theta}=\frac{\pi\beta z_+X(r_1,r_2)}{2\{K(k)Y(r_1,r_2)+a^2z_+^2\left[K(k)-E(k)\right]X(r_1,r_2)\}}, \tag{A.12}$$

$$\frac{\partial H}{\partial J_\phi}=\frac{K(k)Z(r_1,r_2)+L_z[\Pi(z_-^2,k)-K(k)]X(r_1,r_2)}{K(k)Y(r_1,r_2)+a^2z_+^2[K(k)-E(k)]X(r_1,r_2)}. \tag{A.13}$$

The validity of the above solution depends on the Jacobian matrix $\mathsf{D}f$ being non-singular. In fact, $\det\mathsf{D}f\neq 0$ if and only if

$$\det\mathsf{D}(f^{-1})=-\frac{\partial Q}{\partial J_r}\cdot\frac{\partial H}{\partial J_r}=2a^2z_+^2\left[1-\frac{E(k)}{K(k)}\right]\left(\frac{\partial H}{\partial J_r}\right)^2\neq 0, \tag{A.14}$$

as is easily shown by using the second equation of system (A.3). The above condition is fulfilled whenever $k>0$, i.e., if the orbit is non-equatorial. Since $Q=0$ for equatorial orbits, this is immediately clear from the above systems of equations as well. Nevertheless, the frequencies of equatorial orbits can be calculated by taking the limit $k\to 0$ of the general expressions for $k>0$ and, as demonstrated in Section 4.2, one finds agreement with the results obtained from a direct calculation based on the equations of motion.

Appendix B. Calculation of the constants of motion

In order to find the constants of motion E, L_z and Q for given orbital parameters p, e and θ_-, one has to solve the following set of equations:

$$\frac{\mathrm{d}r}{\mathrm{d}\tau} = 0 \qquad \Longleftrightarrow \qquad R(r) = 0, \tag{B.1}$$

$$\frac{\mathrm{d}\theta}{\mathrm{d}\tau} = 0 \qquad \Longleftrightarrow \qquad \Theta(\theta) = 0, \tag{B.2}$$

where $R(r)$ and $\Theta(\theta)$ are given by equations (7) and (8). The roots of these equations correspond to the turning points of, respectively, radial and polar motion, i.e.,we can substitute $r = r_\mathrm{p} = pM/(1+e)$ or $r = r_\mathrm{a} = pM/(1-e)$ in the first equation and $\theta = \theta_-$ in the second equation. Thus, a system of three equations is obtained which can be solved for the three unknown constants of motion, provided that $e \neq 0$. In the case $e = 0$, on the other hand, we have got to make use of the supplemental constraint $R'(r_0) = 0$, in addition to $R(r_0) = 0$ and $\Theta(\theta_-) = 0$, because circular orbits occur if both $R(r)$ and its radial gradient $R'(r)$ vanish at the same point, $r = r_0$.

Introducing the dimensionless quantities

$$\tilde{a} = \frac{a}{M}, \quad \tilde{E} = \frac{E}{\mu}, \quad \tilde{L}_z = \frac{L_z}{\mu M}, \quad \tilde{Q} = \frac{Q}{\mu^2 M^2}, \tag{B.3}$$

we can actually set up a completely scale-invariant formulation of the problem. Hence, it is sufficient to calculate $\tilde{E}$, $\tilde{L}_z$ and $\tilde{Q}$ only once for given orbital parameters. The corresponding physical values for a particular black hole of mass M and a particle of mass μ can then be inferred from the above definitions.

To begin with, we re-arrange $\Theta(\theta_-) = 0$ in order to express Carter's constant $\tilde{Q}$ in terms of θ_-, $\tilde{E}$ and $\tilde{L}_z$,

$$\tilde{Q} = z_-^2 \left[\tilde{a}^2 (1 - \tilde{E}^2) + \frac{\tilde{L}_z^2}{1 - z_-^2} \right], \tag{B.4}$$

where $z_- = \cos\theta_-$. If this expression is substituted for $\tilde{Q}$ in the equation $\tilde{R}(\tilde{r}) = \mu^{-2} M^{-4} R(r) = 0$, we obtain

$$\tilde{R}(\tilde{r}) = f(r)\tilde{E}^2 - 2g(r)\tilde{E}\tilde{L}_z - h(r)\tilde{L}_z^2 - d(r), \tag{B.5}$$

where

$$f(r) = \tilde{r}^4 + \tilde{a}^2 \left[\tilde{r}(\tilde{r}+2) + z_-^2 \tilde{\Delta} \right], \tag{B.6}$$

$$g(r) = 2\tilde{a}\tilde{r}, \tag{B.7}$$

$$h(r) = \tilde{r}(\tilde{r}-2) + \frac{z_-^2}{1 - z_-^2}\tilde{\Delta}, \tag{B.8}$$

$$d(r) = (\tilde{r}^2 + \tilde{a}^2 z_-^2)\tilde{\Delta}, \tag{B.9}$$

and $\tilde{\Delta} = \tilde{r}^2 - 2\tilde{r} + \tilde{a}^2$. Substitution of $\tilde{Q}$ in $\tilde{R}'(r) = 0$ yields

$$\frac{\mathrm{d}\tilde{R}}{\mathrm{d}\tilde{r}} = f'(r)\tilde{E}^2 - 2g'(r)\tilde{E}\tilde{L}_z - h'(r)\tilde{L}_z^2 - d'(r), \tag{B.10}$$

where

$$f'(r) = 4\tilde{r}^3 + 2\tilde{a}^2\left[(1+z_-^2)\tilde{r} + (1-z_-^2)\right], \tag{B.11}$$

$$g'(r) = 2\tilde{a}, \tag{B.12}$$

$$h'(r) = \frac{2(\tilde{r}-1)}{1-\tilde{z}_-^2}, \tag{B.13}$$

$$d'(r) = 2\,(2\tilde{r}-3)\,\tilde{r}^2 + 2\tilde{a}^2\left[(1+z_-^2)\tilde{r} - z_-^2\right]. \tag{B.14}$$

Thus, we shall define the following coefficients in terms of the orbital parameters:

$$(f_1, g_1, h_1, d_1) = \begin{cases} (f(\tilde{r}_{\rm p}), g(\tilde{r}_{\rm p}), h(\tilde{r}_{\rm p}), d(\tilde{r}_{\rm p})) & \text{if } e > 0, \\ (f(\tilde{r}_0), g(\tilde{r}_0), h(\tilde{r}_0), d(\tilde{r}_0)) & \text{if } e = 0, \end{cases} \tag{B.15}$$

$$(f_2, g_2, h_2, d_2) = \begin{cases} (f(\tilde{r}_{\rm a}), g(\tilde{r}_{\rm a}), h(\tilde{r}_{\rm a}), d(\tilde{r}_{\rm a})) & \text{if } e > 0, \\ (f'(\tilde{r}_0), g'(\tilde{r}_0), h'(\tilde{r}_0), d'(\tilde{r}_0)) & \text{if } e = 0. \end{cases} \tag{B.16}$$

Eliminating $\tilde{L}_z$ from the set of quadratic equations,

$$\left.\begin{aligned} f_1\tilde{E}^2 - 2g_1\tilde{E}\tilde{L}_z - h_1\tilde{L}_z^2 - d_1 &= 0, \\ f_2\tilde{E}^2 - 2g_2\tilde{E}\tilde{L}_z - h_2\tilde{L}_z^2 - d_2 &= 0, \end{aligned}\right\} \tag{B.17}$$

a quadratic equation in $\tilde{E}^2$ is obtained,

$$(\rho^2 + 4\eta\sigma)\tilde{E}^4 - 2(\kappa\rho + 2\epsilon\sigma)\tilde{E}^2 + \kappa^2 = 0, \tag{B.18}$$

where ϵ, η, κ, ρ and σ are defined by the following 2×2 determinants:

$$\kappa = \begin{vmatrix} d_1 & h_1 \\ d_2 & h_2 \end{vmatrix}, \qquad \epsilon = \begin{vmatrix} d_1 & g_1 \\ d_2 & g_2 \end{vmatrix}, \tag{B.19}$$

$$\rho = \begin{vmatrix} f_1 & h_1 \\ f_2 & h_2 \end{vmatrix}, \qquad \eta = \begin{vmatrix} f_1 & g_1 \\ f_2 & g_2 \end{vmatrix}, \tag{B.20}$$

$$\sigma = \begin{vmatrix} g_1 & h_1 \\ g_2 & h_2 \end{vmatrix}. \tag{B.21}$$

The roots of equation (B.18) are given by

$$\tilde{E}_\pm^2 = \frac{\kappa\rho + 2\epsilon\sigma \pm 2\sqrt{\sigma(\sigma\epsilon^2 + \rho\epsilon\kappa - \eta\kappa^2)}}{\rho^2 + 4\eta\sigma}. \tag{B.22}$$

Actually, these roots can be calculated regardless of whether the corresponding orbits are of the first or the second kind, i.e., if they are stable or plunging. There is a simple criterion, however, which can be used to determine the kind of the orbit: Let the *potential* of the radial component of motion be defined by

$$\left(\frac{\mathrm{d}r}{\mathrm{d}\tau}\right)^2 = \tilde{E}^2 - \tilde{V}^2. \tag{B.23}$$

According to equation (10), $\tilde{V} = (\tilde{E}^2 - \tilde{R}/\tilde{\Sigma}^2)^{1/2}$ and the radial domain of motion is subject to the constraint $\tilde{V} \le \tilde{E}$. Hence, the orbit is *stable* if

$$\frac{\partial\tilde{V}}{\partial\tilde{r}}(\tilde{r}_{\rm p}) < 0. \tag{B.24}$$

If the gradient of the potential vanishes at $r = r_\mathrm{p}$, the potential has a local maximum at the perihelion radius and, in that case, the orbit is *marginally stable*.

Substituting $\pm\sqrt{E_\pm^2}$ into the system (B.17) and solving for L_z, yields four roots which satisfy both equations. Therefore, we obtain *four* possible solutions for the constants of motion with given orbital parameters, namely, $(-E^{(\mathrm{r})}, -L_z^{(\mathrm{r})}, Q^{(\mathrm{r})})$, $(-E^{(\mathrm{p})}, -L_z^{(\mathrm{p})}, Q^{(\mathrm{p})})$, $(E^{(\mathrm{p})}, L_z^{(\mathrm{p})}, Q^{(\mathrm{p})})$ and $(E^{(\mathrm{r})}, L_z^{(\mathrm{r})}, Q^{(\mathrm{r})})$, where $E^{(\mathrm{p})} = E_-$ is the *lower* and $E^{(\mathrm{r})} = E_+$ the *higher* energy given by equation (B.22). Numerical evaluation shows that $L_z^{(\mathrm{p})} > L_z^{(\mathrm{r})}$. In the first case, the particle has higher binding energy and co-revolves with the rotation of the black hole. Such orbits are called *prograde*. In the second case, the particle has lower binding energy and usually counter-revolves, except for some orbits close to the horizon of rapidly rotating black holes. These orbits are called *retrograde*. The solutions with negative energies correspond to the respective motions under time reversal.

Finally, we consider the *Newtonian* limits of the constants of motion which are asymptotically approached as $p \to \infty$. Using the formulas for the potential energy and angular momentum of Keplerian orbits in geometric units [3],

$$\tilde{U} = -\frac{1-e^2}{2p}, \qquad \tilde{L} = p^{1/2}, \tag{B.25}$$

and noting that $L_z = L\cos(\pi/2 - \theta_-) = L\sin\theta_- = L(1-z_-^2)^{1/2}$, we obtain the following asymptotic expressions for the constants of motion:♯

$$1 - \tilde{E}^2 \simeq \frac{1-e^2}{p}, \qquad \tilde{L}_z^2 \simeq (1-z_-^2)p, \qquad \tilde{Q} \simeq z_-^2 p. \tag{B.26}$$

Since $\tilde{Q} + \tilde{L}_z^2 \simeq \tilde{L}^2$, Carter's constant in the Newtonian limit is the squared angular momentum component parallel to the equatorial plane.

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♯ The constraints $\tilde{R}(\tilde{r}_\mathrm{p}) = 0$ and $\tilde{R}(\tilde{r}_\mathrm{a}) = 0$ are fulfilled asymptotically in the highest power for non-circular orbits, and $\tilde{R}(\tilde{r}_0) = 0$ is satisfied down to $\mathrm{O}(\tilde{r}^{5/2})$ if $e = 0$.

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